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THE CONDITIONS FOR THE SOLUTIONS IN ELECTROMAGNETOELASTICITY TO BE EQUAL TO ZERO*

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The quasistationary antiplane deformation of a cylinder and the twisting of a solid of rotation in conjunction with an electric or magnetic field for non-linear materials are considered. The conditions which guarantee that the stresses, displacements, induction and potential are zero for any boundary conditions are analysed.

For the quasistationary antiplane deformation of a cylinder of circular cross-section made of a linear uniform transversely isotropic piezoelectric material with an unloaded contour, the specification on part of the contour of a constant electric potential, and on part of the contour of the conditions for matching in a vacuum, and also for certain other problems of the same type, it is well-known /1/ that the mechanical stresses are zero everywhere, while the deformations and displacements are proportional to the strength and potential of the electric field. Below we analyse the presence of such properties in a certain class of problems and some allied problems.

1. Formulation of the problem. Consider the quasistationary antiplane deformation of a cylinder (the derivatives with respect to time are zero), in conjunction with a plane electric field for a non-conducting neutral piezoelectric material. On the basis of a well-known analogy, all the results derived later also hold for the twisting of a solid of rotation in conjunction with an axisymmetrical electric field. For brevity we will only consider the antiplane deformation of a cylinder. The transverse cross-section of the cylinder is singly-connected or multiply-connected and is arbitrary.

Suppose x_1, x_2 and x_3 are orthonormalized coordinates, and the x_3 axis is parallel to the generatrix. Further $i = 1, 2$; the notation is that generally used. Suppose the material is such that a state is possible for which only $u_3 = u$; $\gamma_3 = \gamma_i$; $\sigma_{i3} = \sigma_i$; φ; E_i; D_i are non-zero (and are independent of x_3), i.e., it is possible to consider the antiplane deformation of the cylinder in conjunction with a plane electric field (/1-4/ etc.).

We will write the equations of the problem as follows:

$$\sigma_{i,i} = 0;\ \gamma_i = u_{,i};\ \ D_{i,i} = 0;\ \ E_i = \varphi_{,i} \tag{1.1}$$

(φ differs in sign from that usually employed).

We will represent the contour Γ in the form $\Gamma = \Gamma_\sigma + \Gamma_u = \Gamma_D + \Gamma_\varphi$. On Γ we specify mechanical and, simultaneously, electrical boundary conditions as follows:

$$\Gamma_\sigma\colon\ \sigma_i n_i = s_\sigma;\ \ \Gamma_u\colon\ u = s_u \tag{1.2}$$

$$\Gamma_D\colon\ D_i n_i = s_D;\ \ \Gamma_\varphi\colon\ \varphi = s_\varphi \tag{1.3}$$

The case when arbitrary additive constants appear in u and φ is not specially stipulated, and these constants are fixed in a trivial way.

In the antiplane deformation of a cylinder we will consider two classes of non-linear anisotropic non-uniform materials (bodies) - which we will call A and B. Suppose class A is

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described by the equation of state (already referred to the x_1, x_2 plane)

$$\gamma = f_\gamma(x, \sigma) + \alpha E, \quad D = f_D(x, E) + \alpha\sigma \tag{1.4}$$

and class B - similar to the reduced equation of state

$$E = f_E(x, D) + \beta\gamma, \quad \sigma = f_\sigma(x, \gamma) + \beta D \tag{1.5}$$

In (1.4) and (1.5) $x = \{x_1, x_2\}$, $\gamma = \{\gamma_1, \gamma_2\}, \ldots, f_\gamma = \{f_{\gamma 1}, f_{\gamma 2}\}, \ldots$; α, β are scalar constants which do not depend on x_1, x_2. We will understand henceforth by non-linearity, anisotropy and isotropy, non-uniformity etc., properties already referred to the x_1, x_2 plane of the equation of state (for example, a material that is transversely isotropic to x_3 will henceforth be called isotropic).

For brevity we will impose weak constraints on classes A and B. We will begin with class A. We will require that each material, and also materials "associated" with it, preserve the types of boundary conditions, traditional for appropriate linear isotropic uniform materials, i.e. type (1.2) and (1.3), type (1.2), and type (1.3).

We mean by the preservation of type here, in particular, the existence and uniqueness of the solution (not necessarily in the initial class of functions), and the equality of all the required functions to zero if and only if all the quantities specified for this type of boundary-value problem are zero.

A similar constraint is imposed on class B. We can assume that A and B are fairly wide classes of non-linear anisotropic non-uniform materials (the set of "usual" linear isotropic uniform materials belong to the intersection of classes A and B).

2. *Simplification of the solution.* For classes A and B, for certain combinations of paths $\Gamma_\sigma, \Gamma_u, \Gamma_D, \Gamma_\varphi$ it is possible to facilitate the process of solution which consists of replacing the initial connected problem by two unconnected ones. We will write this combination for class A, introducing at the same time some notation, and doing the same for class B:

$$\Gamma_u \subseteq \Gamma_\varphi, \quad s_u^* = s_u - \alpha s_\varphi, \quad s_D^* = s_D - \alpha s_\sigma \tag{2.1}$$

$$u^* = u - \alpha\varphi, \quad \gamma^* = \gamma - \alpha E, \quad D^* = D - \alpha\sigma$$

$$\Gamma_\varphi \subseteq \Gamma_u, \quad s_\varphi^* = s_\varphi - \beta s_u, \quad s_\sigma^* = s_\sigma - \beta s_D$$

$$\varphi^* = \varphi - \beta u, \quad E^* = E - \beta\gamma, \quad \sigma^* = \sigma - \beta D \tag{2.2}$$

We will consider how the simplifications affect situations (2.1) and (2.2). We will begin with class A. For a material of class A in situation (2.1), instead of solving the initial problem we can solve in parallel (independently) the truly elastic problem (for boundary conditions s_u^*, s_σ and the elastic "associated" material) and the purely electrical problem (s_φ, s_D^*, the corresponding material), and then from (2.1) we can obtain the remaining components of the solution. We will write this assertion, and also the corresponding assertion for class B in situation (2.2) in the form of the following scheme (the sign $\Rightarrow$ denotes priority):

$$u^*, \gamma^*, \sigma, \varphi, E, D^* \Rightarrow u, \gamma, D \tag{2.3}$$

$$\varphi^*, E^*, D, u, \gamma, \sigma^* \Rightarrow \varphi, E, \sigma \tag{2.4}$$

Since linear uniform isotropic materials belong to the intersection of classes A and B, for these we have that (2.3) follows from (2.1), and (2.4) follows from (2.2), and when $\Gamma_u = \Gamma_\varphi$ both algorithms (2.3) and (2.4) are applicable.

3. *Equating the solutions to zero.* A consequence of the above is a family of theorems on the equality of the solutions to zero for classes of materials A and B.

We will begin with class A. For this class, in situation (2.1) two assertions hold. The first is as follows. If the functions s_σ, s_u^*, specified on Γ are equal to zero, then everywhere in the region occupied by the body

$$\sigma = 0, \gamma = \alpha E, u = \alpha\varphi$$

for any boundary conditions of the electric problem s_D, s_φ, which may be solvable for the electric "associated" material. We will write this assertion, the second assertion for class A, and two assertions for class B for situation (2.2) in the form of the following schemes:

$$(s_\sigma, s_u^* = 0) \to (\forall s_D, s_\varphi: \sigma = 0; \gamma = \alpha E; u = \alpha\varphi) \tag{3.1}$$

$$(s_D^*, s_\varphi = 0) \to (\forall s_\sigma, s_u: D = \alpha\sigma; E = 0; \varphi = 0)$$

$$(s_\sigma^*, s_u = 0) \to (\forall s_D, s_\varphi: \sigma = \beta D; \gamma = 0; u = 0) \tag{3.2}$$

$$(s_D, s_\varphi^* = 0) \to (\forall s_\sigma, s_u: D = 0; E = \beta\gamma; \varphi = \beta u)$$

For linear isotropic uniform materials when situations (2.1) and (2.2) are identical, both schemes (3.1) and (3.2) hold.

Assertions (3.1) and (3.2) are written for a general form of boundary conditions (1.2) and (1.3) of the mixed problem. In the case of so-called "fundamental boundary-value problems, the formulation of the theorems becomes more laconic.

We note an obvious situation. The analogy between the problems of the antiplane deformation of a cylinder and the equations of state (1.4) and (1.5) (for $f_\gamma \leftrightarrow f_E, f_\sigma \leftrightarrow f_D, \alpha \leftrightarrow \beta$) is a special case of a more general analogy between problems of the antiplane deformation of a cylinder and any equations of state $F_\mu(x, \gamma, \sigma, E, D) = 0$ $(\mu = 1, \ldots, 4)$ and $F_\mu(x, E, D, \gamma, \sigma) = 0$.

4. Applications. At the present time, as we know, piezoelectric properties of different types of materials are used (single crystals, ceramics, polymers etc.). Some of these materials belong to class *A*.

We will give an example. Suppose we have an infinite ceramic cylinder with a piecewise-uniform transverse cross-section, i.e. a body consisting of several infinite cylinders made of different ceramic materials and ideally connected to one another over the cylindrical surfaces. The transverse cross-section and its division into fragment-subregions are arbitrary. The characteristics of the ceramic depend, as we know, on the degree of preliminary polarization along the x_3 axis, and hence to make cylinder-composites the degree of polarization can be taken to be such that for all the materials the value of α is the same. After connecting the cylinders we obtain a body of class *A*.

As another example consider piezoelectric materials made of reinforced (composite) ceramic and polymers (there are many publications on this problem). By varying the parameters of these media in a standard way (the concentration, the direction and material of the fibres, the material of the matrix etc.), we can obtain anisotropic uniform bodies of class *A*. We will not consider the examples in more detail here, since the main purpose of the present communication is solely to state the fact that for any non-linearity, anisotropy and non-uniformity of the material or the cross-section (and also the form of the cross-section itself) (within the framework of class *A*) in the problem of antiplane deformation of a cylinder one obtains the Parton effect /1/ (this relates to existing materials and materials which can be created). The equality of the mechanical stresses to zero in the construction is in a certain sense a rational characteristic, and hence the above description can be used in problems of optimization. Similar considerations relate to materials of class *B*.

Notes.1. The conditions for splitting the problem and the equality of the solutions to zero given here are sufficient; their necessity has not been discussed.

2. The discussion can be extended to the case of a magnetic field in a piezomagnetic material.

3. As pointed out above, all the results for the antiplane deformation of a cylinder can be generalized to the case of the twisting of a solid of rotation in conjunction with an axisymmetrical electric (or magnetic) field. The solid of rotation can have arbitrary non-uniformity, anisotropy and non-linearity (in the framework of the equations of state, which are (1.4) and (1.5)) and a singly connected or multiply connected arbitrary meridian semicross-section.

4. The transverse cross-section of the cylinder or the meridian semicross-section of the solid of rotation may contain curvilinear cuts (cracks), at the edges of which fractures *u* are specified (which generate Somigliani dislocations, and in a special case Volterra dislocations) and discontinuities of the electric or magnetic potentials.

5. Eqs.(1.4)-(3.2) can be generalized to plasticity, creep and corresponding electric (or magnetic) effects. For brevity we will only consider the class of viscoelectroelastic materials in the following. We replace in (1.4) and (1.5) the non-linear anisotropic functions by non-linear anisotropic operators $f_{\gamma i}[x, t, \tau, \sigma(\tau)]_{\tau=-\infty}^{\tau=t}, \ldots$, and the constants α, β by linear operators

$$\int_{-\infty}^{t} \alpha(t, \tau) E_i(\tau)\, d\tau, \ldots$$

Suppose the equations of mechanostatics and electrostatics hold and the materials are so "good" that they do not violate the type of initial-boundary value problem which is inherent in the corresponding linear isotropic uniform materials. Using the notation

$$\gamma_i^*(t) = \gamma_i(t) - \int_{-\infty}^{t} \alpha(t, \tau) E_i(\tau)\, d\tau, \ldots$$

we can obtain, for situations analogous to (2.1) and (2.2), relations similar to (2.3), (2.4), (3.1) and (3.2).

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AN EFFICIENT TECHNIQUE FOR SOLVING A CLASS OF INFINITE SYSTEMS IN CONTACT PROBLEMS IN THE THEORY OF ELASTICITY*

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In this paper we illustrate the efficiency of a method (see, e.g. /1, 2/) of solving infinite systems of linear algebraic equations of the first kind with singular coefficient matrices, to which many problems in the theory of elasticity and mathematical physics with mixed boundary conditions (see e.g. /3-9/) reduce. The method is based on a knowledge of the behaviour of the solution of the system for large numbers, which may be determined from an analysis of the behaviour of the initial problems at particular points. This enables us to reduce an infinite system to and efficiently-solvable finite system. The method does not require the factorization of functions, it enables us to find the principal component of the solution of infinite systems and also to find explicit particular solutions of the problem at points where the boundary conditions change. This method imposes practically no restrictions on the problem parameters and the computation of the solution does not require large amounts of computer time.

1. Problems in the theory of elasticity with mixed boundary conditions may be reduced using the methods of operational calculus for semi-infinite and bounded regions (strips, layers, cylinders, wedges, cones, rectangles, circular plates, rings, etc.) to the solution of pairs (triples, etc.) of integral equations or series equations.

In particular, we consider a triple series equation /3, 4/ of the form

$$\sum_{k=0}^{\infty} Q_k K(u_k)\, y(u_k, x) = f(x) \quad (c \leqslant x \leqslant a) \tag{1.1}$$
$$\sum_{k=0}^{\infty} Q_k y(u_k, x) = 0 \quad (d \leqslant x \leqslant c,\ a \leqslant x \leqslant b)$$

Here Q_k are the desired variables, $y(u_k, x)$ and u_k are (respectively) a system of eigenfunctions and eigennumbers of a Sturm-Liouville problem for a second-order differential equation in a finite interval (see /3-4/), the nature of the function $K(u)$ is also described in /3-4/.

In special cases of this problem, associated with a specific coordinate system, the functions $y(u_k, x)$ are trigonometric functions, Bessel functions, Legendre functions or other

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